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1988 J. Phys. A: Math. Gen. 21 L277

(http://iopscience.iop.org/0305-4470/21/5/002)

J. Phys. A: Math. Gen. 21 (1988) L277-L281. Printed in the UK

LETTER TO THE EDITOR

On the Hamiltonian structure and generalised Lie–Bäcklund symmetries of Langmuir solitons

Swapna Roy and A Roy Chowdhury

High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta 700 032, India.

Received 14 October 1987

Abstract. The structure of generalised Lie-Bäcklund symmetries for the coupled equations of Langmuir solitons are analysed in detail. The form of these symmetries, when compared with those of the conservation laws, yields the first symplectic form. The spectral gradient method is then seen to generate the recursion operator for these symmetries which, on factorisation, leads immediately to the second Hamiltonian structure. The same recursion operator, when used along with the (x, t)-dependent symmetries, yields a new class of generalised symmetries for the equations under consideration. Lastly it is observed that these symmetries are in involution with respect to a Jacobi bracket.

Lie-Bäcklund symmetries have played an important role in the analysis of non-linear partial differential equations [1]. Exhaustive studies were done in the papers of Fokas [2], Fuchssteiner [3], Vinogradov [4] and others. The basic class of equations usually studied is the scalar single-component equation, but occasionally some coupled cases were also considered [5]. Here we have found the full class of Lie-Bäcklund symmetries for the case of Langmuir solitons, governed by a coupled system of a complex equation for the electric field $E(x, t)$ and a real equation for the electron density n. These equations can be written in the form

$$\begin{aligned} &\mathrm{i}E_t+\tfrac{1}{2}E_{xx}-nE=0 \\ &n_t+n_x+|E|^2_x=0. \end{aligned} \tag{1}$$

Let us set $\phi(x, t)=E(x, t)\exp[-\mathrm{i}(t/2-x)]$. Then (1) reduces to

$$\begin{aligned} &\mathrm{i}\phi_t+\mathrm{i}\phi_x+\tfrac{1}{2}\phi_{xx}-n\phi=0 \\ &n_t+n_x+|\phi|^2_x=0. \end{aligned} \tag{2}$$

If we now make the following change of variable:

$$\begin{aligned} &x-t=\gamma x' \qquad t=\varepsilon t' \\ &\phi'=\alpha A \qquad n=\beta B \end{aligned} \tag{3}$$

with $\gamma=2$, $\beta=2$, $\varepsilon=-1$, $\alpha=2\mathrm{i}$, then we get

$$\begin{aligned} &\mathrm{i}A_t-2A_{xx}=-2AB \\ &B_t=-4(AA^*)_x \end{aligned} \tag{4}$$

where we have changed the primed variables to ordinary ones. In the following we will observe that it is convenient to use equation (4) instead of (1).

0305-4470/88/050277+05$02.50

To obtain the conserved quantities we consider an approach advocated by Chen and Liu instead of using the Lax pair proposed in [6]. The linearised equation corresponding to (4) is

$$\begin{pmatrix} A_1 \\ A_1^* \\ B_1 \end{pmatrix}_t = \begin{bmatrix} -2\mathrm{i}\partial_x^2+2\mathrm{i}B & 0 & 2\mathrm{i}A \\ 0 & 2\mathrm{i}\partial_x^2-2\mathrm{i}B & -2\mathrm{i}A^* \\ -4\partial_x A^* & -4\partial_x A & 0 \end{bmatrix} \begin{pmatrix} A_1 \\ A_1^* \\ B_1 \end{pmatrix}. \tag{5}$$

Then the adjoint of equation (5) can be written as

$$\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_1^* \\ \tilde{B}_1 \end{pmatrix}_t + \begin{bmatrix} -2\mathrm{i}\partial_x^2+2\mathrm{i}B & 0 & 4A^*\partial_x \\ 0 & 2\mathrm{i}\partial_x^2-2\mathrm{i}B & 4A\partial_x \\ 2\mathrm{i}A & -2\mathrm{i}A^* & 0 \end{bmatrix} \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_1^* \\ \tilde{B}_1 \end{pmatrix} - 0. \tag{6}$$

We search for a solution in the form

$$\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_1^* \\ \tilde{B}_1 \end{pmatrix} = \begin{pmatrix} 1 \\ b(k) \\ c(k) \end{pmatrix} \exp\left(Kx + 2\mathrm{i}K^2 t + \int_x^\alpha \sigma \, \mathrm{d}x \right) \tag{7}$$

and assume σ, $b(k)$, $c(k)$ to be expandable in the form:

$$\sigma = \sum_{j=0}^{\alpha} \sigma_j k^{-j} \qquad b(k) = \sum_{j=0}^{\alpha} b_j k^{-j} \qquad c(k) = \sum_{j=0}^{\alpha} c_j k^{-j}$$

which leads to a coupled set of recursion relations for σ_j, b_j, c_j, which can be solved in stages to yield

$$\begin{aligned}
&\sigma_0 = 0 \\
&\sigma_1 = \tfrac{1}{2}B \\
&4\mathrm{i}\sigma_2 = -\mathrm{i}B_x - 6AA^* \\
&4\mathrm{i}\sigma_3 = \tfrac{1}{2}\mathrm{i}B_{xx} - \tfrac{1}{2}\mathrm{i}B^2 + A_x^* A + 3A^* A_x \\
&4\mathrm{i}\sigma_4 = -\tfrac{15}{2}A_{xx}A^* - 2A_x A_z^* - \tfrac{1}{2}AA_{xx}^* + 6AA^*B + \mathrm{i}BB_x - \tfrac{1}{4}\mathrm{i}B_{xxx} \\
&4\mathrm{i}\sigma_5 = \tfrac{15}{4}A_{xxx}A^* + \tfrac{11}{4}A_{xx}A_x^* + \tfrac{5}{4}A_{xx}^*A_x + \tfrac{1}{4}A_{xxx}^*A - \tfrac{9}{2}A_x A^* B - \tfrac{3}{2}A_x^* AB \\
&\qquad - 3AA^*B_x + \tfrac{3}{2}\mathrm{i}(AA^*)^2 - \tfrac{3}{4}\mathrm{i}(BB_{xx}) + \tfrac{1}{4}\mathrm{i}B^3 - \tfrac{5}{8}\mathrm{i}B_x^2 + \tfrac{1}{8}\mathrm{i}B_{xxx}.
\end{aligned} \tag{8}$$

The conserved quantities are nothing other than the variational derivatives $\delta\sigma_i/\delta A$, $\delta\sigma_i/\delta A^*$, $\delta\sigma_i/\delta B$. One can actually find the infinite class of σ_i from (6) and (7).

We now proceed directly with the linearised equation (5) instead of its adjoint and try to find solutions of A_1, A_1^*, B_1 in the jet space (A_i, B_i; $A_i = \partial^i A/\partial x^i$, $B_i = \partial^i B/\partial x^i$). The total time and space derivative operators are written as

$$\begin{aligned}
D_x &= \frac{\delta}{\delta x} + \sum A_{i+1}\frac{\delta}{\delta A_i} + \sum A_{i+1}^*\frac{\delta}{\delta A_i^*} + \sum B_{i+1}\frac{\delta}{\delta B_i} \\
D_t &= \frac{\delta}{\delta t} + \sum A_{it}\frac{\delta}{\delta A_i} + \sum A_{it}^*\frac{\delta}{\delta A_i^*} + \sum B_{it}\frac{\delta}{\delta B_i}.
\end{aligned} \tag{9}$$

Then a laborious computation leads to the following set of solutions for the linearised fields, A_1, A_1^*, B_1:

$$
\begin{aligned}
&\text{(i)} && A_1^1 = -2\mathrm{i}A \qquad A_{1*}^1 = \mathrm{i}A^* \qquad B_1^1 = 0 \\
&\text{(ii)} && A_1^2 = \tfrac{2}{3}\mathrm{i}A_x \qquad A_{1*}^2 = \tfrac{2}{3}\mathrm{i}A_x^* \qquad B_1^2 = \tfrac{2}{3}\mathrm{i}B_x \\
&\text{(iii)} && A_1^3 = -2\mathrm{i}(A_{xx} - AB) \qquad A_{1*}^3 = 2\mathrm{i}(A_{xx}^* - A^*B) \\
& && B_1^3 = -4(AA^*)_x \\
&\text{(iv)} && A_1^4 = \tfrac{2}{3}\mathrm{i}A_{xxx} - \mathrm{i}A_xB - \tfrac{1}{2}\mathrm{i}AB_x - A^2A^* \\
& && A_{1*}^4 = \tfrac{2}{3}\mathrm{i}A_{xxx}^* - \mathrm{i}A_x^*B - \tfrac{1}{2}\mathrm{i}A^*B_x - AA^{*2} \\
& && B_1^4 = (A_xA^* - A_x^*A) - \tfrac{1}{2}\mathrm{i}(B^2)_x + \tfrac{1}{6}\mathrm{i}B_{xxx}.
\end{aligned}
\tag{10}
$$

Now an interesting and important observation is that

$$
\begin{pmatrix} A_1^i \\ A_{1*}^i \\ B_1^i \end{pmatrix} = \begin{pmatrix} 0 & -\frac{4}{3} & 0 \\ \frac{4}{3} & 0 & 0 \\ 0 & 0 & -\frac{8}{3}\mathrm{i}\partial_x \end{pmatrix} \begin{pmatrix} \delta\sigma^{i+1}/\delta A \\ \delta\sigma^{i+1}/\delta A^* \\ \delta\sigma^{i+1}/\delta B \end{pmatrix}. \tag{11}
$$

Thus the matrix

$$
J = \begin{pmatrix} 0 & -\frac{4}{3} & 0 \\ \frac{4}{3} & 0 & 0 \\ 0 & 0 & -\frac{8}{3}\mathrm{i}\delta_x \end{pmatrix}
$$

can be considered to give the first symplectic structure of the Langmuir solitons. Indeed, it is easy to see that

$$
\begin{pmatrix} A_t \\ A_t^* \\ B_t \end{pmatrix} = J \begin{pmatrix} \delta\sigma_4/\delta A \\ \delta\sigma_4/\delta A^* \\ \delta\sigma_4/\delta B \end{pmatrix} \tag{12}
$$

which is nothing other than the original equation of motion.

The existence of an infinite number of conservation laws and their corresponding symmetries suggests that it will be possible to find an operator (the recursion operator R) which will generate the whole hierarchy of LB symmetries starting from the lowest one. The most useful technique to construct a recursion operator is to impose the condition that

$$
R\eta^i = \eta^i + 1
$$

where

$$
\eta^i = \begin{pmatrix} A_1^i \\ A_{1*}^i \\ B_1^i \end{pmatrix}. \tag{13}
$$

But some important restrictions that are to be imposed on R are (i) it is not always possible to connect the lowest LB symmetry via R, (ii) it is also not true that the generation via R will always be consecutive. Actually, in our present situation we have observed that R connects only even η, i.e. η^{2n}, and odd η, i.e. η^{2n+1}, separately. Mathematically it amounts to

$$
\eta^{2n+2} = R\eta^{2n} \qquad \eta^{2n+1} = R\eta^{2n-1}. \tag{14}
$$

From the form of the symmetries given in (10) it immediately implies that R possesses the structure

$$R=\begin{pmatrix} \partial_x^2+a & b & c \\ d & \partial_x^2+e & f \\ g & h & \partial_x^2/4+k \end{pmatrix} \tag{15}$$

where a, b, c, d, etc, may contain D, D^{-1}, etc, and the field variables. In this particular case we have seen that it is given as

$$R=\begin{bmatrix} D^2-\frac{3}{2}\mathrm{i}A^*D^{-1}A-B & \frac{3}{2}\mathrm{i}A^*D^{-1}A^* & \frac{3}{2}\mathrm{i}A^*D+\mathrm{i}A_x^* \\ -\frac{3}{2}\mathrm{i}AD^{-1}A & D^2+\frac{3}{2}\mathrm{i}AD^{-1}A^*-B & -\frac{3}{2}\mathrm{i}AD-\mathrm{i}A_x \\ -\frac{1}{2}D^{-1}AD-\frac{1}{4}A & -\frac{1}{2}D^{-1}A^*D-\frac{1}{4}A^* & -\frac{1}{2}B-\frac{1}{2}D^{-1}BD+\frac{1}{4}D^2 \end{bmatrix}. \tag{16}$$

Verification can be made over the form of symmetries given in (10). Until now we have not used the inverse scattering equation for (1). At this stage we may use the spectral gradient method of Fokas [5] to verify (16). The IST equations pertaining to this equation were suggested by Yan-Chow Ma [6]:

$$\begin{aligned} \partial V_1/\partial x &= 3\mathrm{i}\zeta V_1+AV_2+\mathrm{i}BV_3 \\ \partial V_2/\partial x &= 2\mathrm{i}\zeta V_2+A^*V_3 \\ \partial V_3/\partial x &= \mathrm{i}\zeta V_3-\mathrm{i}V_1 \end{aligned} \tag{17}$$

or $\mathrm{d}V/\mathrm{d}x=u(\zeta,x)V\cdot$, ζ being the eigenvalue of the problem. Let us now compute the gradient of ζ and call it G_ζ. If we can show that G_ζ satisfies

$$\psi G_\zeta=\sigma(\zeta)G_\zeta$$

i.e. G_ζ follows an eigenvalue equation. Then G_ζ can be identified with the operator R. In the present case G_ζ turns out to be proportional to $(V\cdot V^*)$, the analogue of the square eigenfunction where V^* is the adjoint of V. It is now a matter of computation to verify that

$$R\begin{pmatrix} -V_2V_1^* \\ -V_3V_2^* \\ -\mathrm{i}V_3V_1^* \end{pmatrix}=-\zeta^2\begin{pmatrix} -V_2V_1^* \\ -V_3V_2^* \\ -\mathrm{i}V_3V_1^* \end{pmatrix} \tag{18}$$

so that the form of recursion operator is verified from two points of view.

It is now very interesting to observe that

$$R=J^{-1}M$$

where

$$M=\begin{bmatrix} -\frac{3}{2}\mathrm{i}AD^{-1}A & D^2+\frac{3}{2}\mathrm{i}AD^{-1}A^*-B & -\frac{3}{2}\mathrm{i}AD-\mathrm{i}A_x \\ -D^2+\frac{3}{2}\mathrm{i}A^*D^{-1}A+B & -\frac{3}{2}\mathrm{i}A^*D^{-1}A^* & -\frac{3}{2}\mathrm{i}A^*D-\mathrm{i}A_x^* \\ -\mathrm{i}AD-\frac{1}{2}\mathrm{i}DA & -\mathrm{i}A^*D-\frac{1}{2}\mathrm{i}DA^* & -\mathrm{i}DB-\mathrm{i}BD+\frac{1}{2}\mathrm{i}D^3 \end{bmatrix}. \tag{19}$$

It then easily follows that

$$\begin{pmatrix} A_t \\ A_t^* \\ B_t \end{pmatrix}=J\begin{pmatrix} \delta\sigma_4/\delta A \\ \delta\sigma_4/\delta A^* \\ \delta\sigma_4/\delta B \end{pmatrix}=M\begin{pmatrix} \delta\sigma_2/\delta A \\ \delta\sigma_2/\delta A^* \\ \delta\sigma_2/\delta B \end{pmatrix} \tag{20}$$

explicitly displaying the bi-Hamiltonian structure associated with the Langmuir soliton equations. So M gives the second symplectic structure.

The involutive character of the symmetries follows from the Jacobi bracket defined via

$$\{\eta, \sigma\} = \ni(\eta)[\sigma] - \ni(\sigma)[\eta] \tag{21}$$

where $[\eta]$ and $[\sigma]$ stand for the symmetry vectors (η_1, η_2, η_3) and $(\sigma_1, \sigma_2, \sigma_3)$ along with

$$\ni(\eta) = \begin{bmatrix} \sum \frac{\delta\eta_1}{\delta A^i} D^i & \sum \frac{\delta\eta_1}{\delta A^{i*}} D^i & \sum \frac{\delta\eta_1}{\delta B^i} D^i \\ \sum \frac{\delta\eta_2}{\delta A_i} D^i & \sum \frac{\delta\eta_2}{\delta A_i^*} D^i & \sum \frac{\delta\eta_2}{\delta B_i} D^i \\ \sum \frac{\delta\eta_3}{\delta A_i} D^i & \sum \frac{\delta\eta_3}{\delta A_i^*} D^i & \sum \frac{\delta\eta_3}{\delta B_i} D^i \end{bmatrix}. \tag{22}$$

The Jacobi bracket was initially introduced by Vinogradov [4] in his study of time-dependent symmetries of the Burger equation. It was also seen to be useful in the general study of non-local and generalised symmetries as discussed by Kosmann-Schwarzbach [7].

Lastly we can mention that a new kind of (x, t)-dependent symmetry can be obtained if a lowest-order (x, t)-dependent symmetry can be found. In the present case we have observed that $\eta = (tA, -tA^*, -\frac{1}{2}\mathrm{i})$ is a symmetry vector, but it is also in involution with the ordinary symmetries via (21).

In our previous analysis we have obtained the structure of symmetries, conservation laws, recursion operator and bi-Hamiltonian structure associated with a coupled system of three equations. However, the analysis never uses the inverse scattering technique except at some stage to verify the form of the recursion operator.

This work was part of the programme sponsored by DST (Government of India) through a Thrust Area Project.

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